

How many edge-colors for almost-rainbow K_5 s?

Elliot Krop*

March 6, 2008

Abstract

Let $f(n)$ be the minimum number of colors necessary to color the edges of K_n so that every path or cycle with four edges is at least three-colored. We show that $f(n) \geq \frac{11}{4}n - \frac{23}{4}$ improving on the bound of Axenovich for the generalized Ramsey number $f(n, 5, 9)$ first studied by Erdős and Gyárfás.

2000 Mathematics Subject Classification: 05A15, 05C38, 05C55

Keywords: Ramsey theory, generalized Ramsey theory, rainbow-coloring, edge-coloring, Erdős problem

1 Introduction

1.1 Definitions

For basic graph theoretic notation and definition see the monograph by Diestel [3]. All graphs G are undirected with the vertex set V and edge set E . We use $|G|$ for $|V|$ and $\|G\|$ for $|E|$. K_n denotes the complete graph on n vertices and $K_{n,m}$ the bipartite graph with n vertices and m vertices in the first and second part, respectively. For any edge (u, v) , let $c(u, v)$ be the color on that edge, and for any vertex v , let $c(v)$ be the set of colors on the edges incident to v . We say that an edge-coloring is *proper* if every pair of incident edges are of different colors.

1.2 Coloring Edges

Given a graph G of order n and integers p, q so that $2 \leq p \leq n$ and $1 \leq q \leq \binom{p}{2}$, call an edge-coloring (p, q) if every $K_p \subseteq K_n$ receives at least q colors on its edges. Let $f(n, p, q)$ be the minimum number of colors in a (p, q) coloring of K_n . This generalization of classical Ramsey functions was first mentioned by P. Erdős in [4] and later studied by Erdős and Gyárfás in [5].

Extending the definition, for any graph G , call an edge coloring (H, q) if every subgraph $H \subseteq G$ receives at least q colors on its edges. Let $f(G, H, q)$ be the minimum colors in an (H, q) coloring of the edges of G . We say that a coloring of H is *almost-rainbow* if $q = \|H\| - 1$, that is, one color is repeated once.

Using the Local Lemma, the authors in [5] were able to produce bounds for $f(n, p, q)$, with several difficult cases unresolved. Among those were $f(n, 4, 3)$ and

*Department of Mathematics and Statistics, Loyola University Chicago, (ekrop@luc.edu).

$f(n, 5, 9)$. In these cases they showed that $f(n, 4, 3) \leq c\sqrt{n}$ and $\frac{4}{3}n \leq f(n, 5, 9) \leq cn^{\frac{3}{2}}$. The authors further mentioned that in this branch of generalized Ramsey theory, finding the orders of magnitude of $f(n, 4, 4)$ and $f(n, 5, 9)$ are “the most interesting open problems, at least to show that the latter is non-linear”. The authors then stated the linearity of said lower bound as Problem 1.

In [7], D.Mubayi showed that

$$f(n, 4, 3) \leq e^{O(\sqrt{n})}$$

and in [8] A.Kostochka and D.Mubayi showed that for some constant c ,

$$f(n, 4, 3) \geq \frac{c \log n}{\log \log \log n}$$

Recently, J.Fox and B. Sudakov in [6], further improved the upper bound to $\frac{\log n}{4000}$.

As for the other case, in [1], M.Axenovich showed that for some constant c ,

$$\frac{1 + \sqrt{5}}{2}n \leq f(n, 5, 9) \leq 2n^{1 + \frac{c}{\sqrt{\log n}}}$$

In that same paper, she remarked that G.Tóth had communicated to her that the lower bound can be improved to $2n - 6$.

In this paper we show that

$$f(n, 5, 9) \geq \frac{11}{4}n - \frac{23}{4}$$

2 The main tool

Let $f(n)$ be the minimum number of colors needed to color the edges of K_n so that every path or cycle with four edges is at least three-colored. By arguments from [1] it is easy to see that $f(n, 5, 9) = f(n)$.

Let $G = K_{2,n}$, for integer $n \geq 2$. Call the vertices in the first part of G u, v . We color the edges of $K_{2,n}$ so that every path or cycle with four edges is at least three-colored. Notice that the minimum number of colors for such a coloring is $f(K_{2,n}, K_{2,3}, 5)$. Indeed, given $K_{2,3}$ with five colors on its six edges, one color must be repeated. Consequently, every path or cycle with four edges can have at most one repeated color, therefore receiving at least three colors. On the other hand, every $K_{2,3}$ containing a path or cycle with four edges colored by three colors, must receive at least five colors.

Lemma 2.1 $f(K_{2,n}, K_{2,3}, 5) = \lceil \frac{3}{2}n \rceil$

Proof.

Suppose the edges of $G = K_{2,n}$ are properly colored so that every $K_{2,4}$ receives at least seven colors. Then for every color $a \in c(u) \cap c(v)$, there exist colors b, c so that $b, c \in (c(u) \cup c(v)) \setminus (c(u) \cap c(v))$. Since there are two colors for every one in $c(u) \cap c(v)$, we can say that

$$|c(u) \cap c(v)| \leq \lfloor \frac{1}{2} |(c(u) \cup c(v)) \setminus (c(u) \cap c(v))| \rfloor \quad (2.1)$$

Applying this inequality to the principle of inclusion-exclusion, we write

$$|c(u) \cup c(v)| = |c(u)| + |c(v)| - |c(u) \cap c(v)| \geq 2n - \frac{1}{3} |c(u) \cup c(v)|$$

Solving for the union we get

$$|c(u) \cup c(v)| \geq \frac{3}{2}n \quad (2.2)$$

Next suppose that the coloring is not proper and let a and b be colors in $c(u)$. If a is used on incident edges adjacent to u and b is used on incident edges adjacent to u , then consider the induced subgraph composed of the vertices incident to any four edges colored by a or b . Notice that the maximum number of colors used on the edges of this subgraph is six. Hence, the maximum number of times that a color can be repeated in $c(u)$ is once.

Assume $a \in c(u)$ is repeated once, no two color in $c(v)$ are the same, and let v_1, v_2 be vertices incident to the edges colored a . Notice that the edge coloring on $H := G \setminus \{v_1, v_2\}$ is proper. Applying the above result for proper colorings, we conclude that the number of colors on the edges of H is at least $\frac{3}{2}(n-2)$. Since $c(v_1, v) \neq c(v_2, v)$, the number of colors on the edges of G is at least $\frac{3}{2}(n-2) + 3 = \frac{3}{2}n$ as desired.

Last, suppose that $a \in c(u)$ is repeated once and $b \in c(v)$ is repeated once. Let v_1, v_2 be vertices incident to the edges colored a and v_3, v_4 be vertices incident to the edges colored b . Notice that the edge coloring on $H := G \setminus v_1, v_2, v_3, v_4$ is proper. Applying the result for proper colorings, we conclude that the number of colors on the edges of H is at least $\frac{3}{2}(n-4)$. Since $c(v_1, v) \neq c(v_2, v)$, and $c(v_3, u) \neq c(v_4, u)$, the number of colors on the edges of G is at least $\frac{3}{2}(n-4) + 6 = \frac{3}{2}n$ concluding the proof of the upper bound.

For the lower bound, we construct an edge-coloring of $G = K_{2,n}$ with $\lceil \frac{3}{2}n \rceil$ colors. Label the vertices of the first part of G u, v and the second part $\{v_1, v_2, \dots, v_n\}$. Let $r = \lfloor \frac{n}{2} \rfloor$. Color the edges $(v_1, u), (v_2, u), \dots, (v_r, u)$ by the colors $1, \dots, r$ and the edges $(v_n, v), (v_{n-1}, v), \dots, (v_{n-r+1}, v)$ by the colors $1, \dots, r$. Color the remaining edges distinctly by the colors not previously used. Let i and j be such that $c(u, v_i) = c(v, v_j)$. Notice that for any $k \in \{1, \dots, n\}$, $\{c(u, v_i), c(u, v_j), c(v, v_i), c(v, v_k)\}$ are pairwise distinct.

Also, $\{c(u, v_i), c(u, v_j), c(v, v_i), c(v, v_k)\}$ are pairwise distinct. Hence every $K_{2,3}$ receives at least five colors.

□

3 The result

Theorem 3.1 $f(n, 5, 9) \geq \frac{11}{4}n - \frac{23}{4}$

Proof.

Consider a $(5, 9)$ edge-coloring of $G = K_n$ using s colors. Using the argument of M. Axenovich [1], we first assume that the coloring is not proper, so there exist incident edges (v_1, v_2) and (v_1, v_3) of the same color. For the coloring to remain $(5, 9)$, all edges of G incident to $\{v_1, v_2, v_3\}$ must be of different colors. Therefore, $s \geq 3n - 7 \geq \frac{11}{4}n - \frac{23}{4}$ for $n \geq 5$.

Next we assume the coloring is not proper. By the pigeonhole principle there exists a color, call it a , used on at least $\binom{n}{2}/s$ edges. Let A be the set of vertices adjacent to edges colored a and choose vertices $u, v \in A$ so that $c(u, v) = a$.

We say that an edge is *in* A if both vertices incident to that edge are in A . Notice that the number of colors on the edges of A incident to $u \geq 2\binom{n}{2}/s - 1$ and the same for v . Notice that we counted $c(u, v)$ twice. Let H be the complete bipartite graph with $\{u, v\}$ vertices in the first part and the vertices of $G \setminus A$ in the second part. Let the edge coloring of H be induced by the edge coloring of G . For any $x \in A$ and $y \in G$, $c(u, x) \neq c(v, y)$, else we produce a two-colored four-edge path. The same reasoning holds for $y \in A$ and $x \in G$. This implies that the colors on the edges of H are distinct from the colors previously counted. Hence we apply Lemma 2.1 to H to obtain

$$s \geq 2\binom{n}{2}/s - 1 + 2\binom{n}{2}/s - 1 - 1 + \frac{3}{2}(n - 2\binom{n}{2}/s) \quad (3.1)$$

Solving for s we obtain the result. □

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